UNIVERSITY OF CINCINNATI

May 23, 1945

I hereby recommend that the thesis prepared under my supervision by Donald Joseph Montgomery

entitled RELATIVISTIC INTERACTION OF ELECTRONS ON THE GENERALIZED QUANTUM ELECTRODYNAMICS

be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy in physics.

Approved by:

Boris Podolsky

S. J. M. Allen

Form 668—G. S. and Ed.—1M—7-37

RELATIVISTIC INTERACTION
OF ELECTRONS
ON THE
GENERALIZED QUANTUM ELECTRODYNAMICS

A dissertation submitted to the

Graduate School of Arts and Sciences
of the University of Cincinnati

in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

1945

by

Donald Joseph Montgomery

Ch.E. University of Cincinnati 1939

TABLE OF CONTENTS

	Page
INTRODUCTION	1
Basic Principles	1
The Generalized Quantum Electrodynamics	3
Formalism	5
WAVE EQUATION FOR A SYSTEM OF PARTICLES	8
Derivation of Wave Equation for a System of Particles	8
Representation of Wave Equation in Functional Formalism	10
Explicit Representation for Field Operators and Functional	13
Application to Wave Equation	14
RELATIVISTIC INTERACTION OF TWO ELECTRONS	16
Wave Function for the Unperturbed System	16
Elimination of ψ_1 and $\tilde{\psi}_1$	17
Calculation of Interaction Matrix Element	19
APPENDIX	21

ACKNOWLEDGMENT

The author is grateful to Dr. Boris Podolsky for his continued encouragement and assistance.

INTRODUCTION

Many of the attempts to provide a general physical theory including gravitational, relativistic, and quantum effects have led to generalized field theories. The method of attack has proceeded from generalizations of the formalism of classical particle dynamics, classical electrodynamics, and particle quantum mechanics. The synthesis and extension of the ideas of the older theories were initiated by Dirac[1], Jordan[2], Heisenberg[3], and Pauli[2,3].

Basic Principles

The procedure[4] is usually to select a Lagrangian L as a function of the field coordinates $\varphi_\alpha = (\underline{A}, i\varphi)$ and their derivatives:

$$L = L\,(\varphi_\alpha,\ \varphi_{\alpha,\beta}\,,\ \varphi_{\alpha,\beta\gamma}\ \ldots.),$$

where the φ_α are functions of the space-time coordinates $x_\alpha = (x_1, x_2, x_3, x_4 = ix_0 = ict)$ and

$$\varphi_{\alpha,\beta} = \frac{\partial \varphi_\alpha}{\partial x_\beta}\,, \qquad \varphi_{\alpha,\beta\gamma} = \frac{\partial \varphi_\alpha}{\partial x_\beta\, \partial x_\gamma}$$

---

1 P. A. M. Dirac, Proc. Roy. Soc. 114, 243, 710 (1927).
2 P. Jordan and W. Pauli, Zeits. f. Physik, 47, 151 (1928).
3 W. Heisenberg and W. Pauli, Zeits. f. Physik, 56, 1 (1929); 59, 169 (1930).
4 See, for example, the review article by W. Pauli, Rev. Mod. Phys. 13, 203 (1941).

The application of the variational principle

$$ic\delta W = \delta\int L\, d\Omega = 0, \quad d\Omega = dx_1 dx_2 dx_3 dx_4 \qquad (1.1)$$

leads to the field equations (the form of which depends on the order of the derivatives in the Lagrangian), when the variables in the Lagrangian are specified and unvaried on the boundaries of the four-dimensional manifold over which the integral is taken. A different application of the variation procedure, where the potentials as well as the boundaries are varied, while the field equations are satisfied throughout the region of integration, as follows:[5]

$$ic\delta W = \int_{\xi_\beta}^{x_\beta + \delta x_\beta} (L + \delta L) d\Omega - \int_{\xi_\beta}^{x_\beta} L\, d\Omega, \qquad (1.2)$$

leads to the determination of the energy-momentum tensor, with the aid of the definition of the total momentum P_μ through

$$\delta W \equiv P_\mu \delta x_\mu,$$

and of the energy-momentum tensor $t_{\mu\nu}$ through

$$P_\mu \equiv \int t_{\mu\nu}\, dS_\nu.$$

Quantization is effected by the Heisenberg-Pauli rule, which is a generalization of the particle quantum mechanical rule

$$p_j q_{j'} - q_{j'} p_j = -\hbar i \delta_{jj'},$$

---

5 The notation and techniques of this section are the same as those used by B. Podolsky and C. Kikuchi, Phys. Rev. 65, 228 (1944). This paper will be designated as GE II.

to a form applicable to field quantities, at a given instant:

$$p(\underline{r})q(\underline{r}') - q(\underline{r}')p(\underline{r}) = -\hbar i\delta(\underline{r} - \underline{r}') \qquad (1.4)$$

or its alternate,

$$\dot{F}(i/\hbar) = [\dot{H}, F]. \qquad (1.5)$$

Here p_j, the particle momentum component conjugate to q_j, is defined in terms of the derivative of the Lagrangian; and $p(\underline{r})$, the field momentum component conjugate to $q(\underline{r})$, any one component of the potential, is defined in terms of functional derivatives of $\bar{L} = \int L\,dV$ (see, for example, GE II, eqns. 1.4, 1.5).

Now the potentials are not uniquely determined by what may be thought of as the observable quantities, the field strengths $\underline{\mathcal{E}}$ and $\underline{\mathcal{H}}$; as a consequence there is some arbitrariness in the potentials which may be reduced by the specification of auxiliary conditions. The choice of conditions is restricted in quantum electrodynamics by requirements imposed by the commutation conditions arising in a q-number theory, as well as by the customary requirements of the c-number electrodynamics. There is usually sufficient latitude, however, to permit a useful choice of auxiliary conditions.

The Generalized Quantum Electrodynamics

In the usual quantum electrodynamics the Lagrangian is restricted to contain derivatives of order not higher than first. Podolsky[6] has postulated a generalized electrodynamics by including second derivatives of the potentials in the Lagrangian. In the non-quantum electrostatic case it was shown that the field and the self-energy of a point charge are everywhere finite; and in the non-quantum relativistic case for space free of charged particles that the solution of the field equations for the potentials consists of two parts, the ordinary waves, corresponding to the usual electrodynamical waves describing photons, and the extraordinary waves, analogous to de Broglie waves for neutral particles.

In GE II Podolsky and Kikuchi extended the theory to include quantum effects, by applying to the Lagrangian containing the continuous set of field coordinates, together with their first and second derivatives, a generalization of the technique suggested by Ostrogradsky[7] for application to a Lagrangian containing a discrete set of particle coordinates, together with their first and second derivatives. The formalism of Fock and Podolsky[8] was used in establishing commutation rules for field strengths, and for the selection of the auxiliary conditions, following the work of Dirac,

---

6 B. Podolsky, Phys. Rev. 62, 68 (1942). This paper may be called GE I.

7 See E. T. Whittaker, Analytical Dynamics (1927), Chapter X.

8 V. Fock and B. Podolsky, Physik. Zeits. Sowjetunion 1, 801 (1932). This paper will be designated as FP.

Fock, and Podolsky[9], and a later paper by Stückelberg[10]. In the fashion of DFP the field equations were derived.

Podolsky and Kikuchi[11], through elimination of the auxiliary conditions in the manner of Fock[12], derived the expression for the electrostatic interaction and the self-energy of point charges in the quantum case (which values turned out to be the same as for the non-quantum case), and also derived the relativistic wave equation for a single particle.

In the present paper the relativistic wave equation for a system of particles is derived. A generalization of Fock's formalism[12] is developed, and applied to obtain the first-order approximation of the relativistic interaction between two electrons, on the generalized quantum electrodynamics. The result contains no infinities.

Formalism

In particle quantum mechanics, the solution of particular problems is often greatly facilitated by choosing specific forms for the operands to be used in conjunction

9 P. Dirac, V. Fock, and B. Podolsky, Physik. Zeits. Sowjetunion 2, 468 (1932). This paper will be designated as DFP.

10 E. C. G. Stückelberg, Helv. Phys. Acta 11, 225 (1938).

11 B. Podolsky and C. Kikuchi, Phys. Rev. 67, 184 (1945). This paper will be designated as GE III.

12 V. Fock, Physik. Zeits. Sowjetunion 6, 425 (1934).

with operator equations. The usual Schroedinger formulation results when the operands are taken as functions in configuration space, and each spatial coordinate operator is represented by the spatial coordinate, each conjugate momentum by a factor times the derivative with respect to the coordinate; for example, the equation

$$p_j q_{j'} - q_{j'} p_j = -\hbar i \delta_{jj'} \qquad (1.3)$$

permits the representation

$$(-i\hbar \partial/\partial q_j) q_{j'} \psi - q_{j'}(-i\hbar \partial/\partial q_j)\psi = -\hbar i \delta_{jj'} \psi \qquad (1.6)$$

where $\psi = \psi(q_j)$.

Fock in a series of papers[13] has developed an analogous formulation for field quantum mechanics. For this case the operands are taken as functionals of the field variables, and each field coordinate operator is represented by the field coordinate, each conjugate momentum by a factor times the functional derivative with respect to the coordinate; for example, the equation

$$p(\underline{r})q(\underline{r}') - q(\underline{r}')p(\underline{r}) = -\hbar i \delta(\underline{r} - \underline{r}') \qquad (1.4)$$

will allow the representation

$$(-i\hbar \delta/\delta q(\underline{r}))q(\underline{r}')\Psi - q(\underline{r}')(-i\hbar \delta/\delta q(\underline{r}))\Psi = -\hbar i \delta(\underline{r} - \underline{r}')\Psi \qquad (1.7)$$

where $\Psi = \Psi[q(\underline{r})]$.

Fock has provided illustrations of the use of this repre-

13 V. Fock, Zeits. f. Physik, 49, 339 (1928); 75, 622 (1932); and reference 10.

sentation in reference 11, and further application of the technique is made in GE III: it furnishes the basis for the computations of the present paper.

WAVE EQUATION FOR A SYSTEM OF PARTICLES

Derivation of Wave Equation for a System of Particles

According to DFP, GE II, and GE III, the Dirac wave equation for a system of particles and field, together with their interaction, is

$$(H_p + \bar{H}_f + H_{pf})\Psi = i\hbar\,\partial\Psi/\partial T \tag{2.1}$$

with

$$\psi = \psi(r_1 \ldots r_n;\ \underline{A}(\underline{k}), \underline{A}^*(\underline{k}), \varphi(\underline{k}), \varphi^*(\underline{k});\ \tilde{\underline{A}}(\underline{k}), \tilde{\underline{A}}^*(\underline{k}), \tilde{\varphi}(\underline{k}), \tilde{\varphi}^*(\underline{k});\ T) \tag{2.2}$$

where

$$H_p = \sum_{s=1}^{n} (c\underline{\alpha}_s \cdot \underline{p}_s + m_s c^2 \beta_s); \tag{2.3}$$

$$\begin{aligned}\bar{H}_f = &\int [\underline{A}^*(\underline{k})\cdot\underline{A}(\underline{k}) - \varphi^*(\underline{k})\varphi(\underline{k}) + \underline{A}(\underline{k})\cdot\underline{A}^*(\underline{k}) \\ &\qquad - \varphi(\underline{k})\varphi^*(\underline{k})]\, k^2\, d\underline{k} \\ &- \int [\tilde{\underline{A}}^*(\underline{k})\cdot\tilde{\underline{A}}(\underline{k}) - \tilde{\varphi}^*(\underline{k})\tilde{\varphi}(\underline{k}) + \tilde{\underline{A}}(\underline{k})\cdot\tilde{\underline{A}}^*(\underline{k}) \\ &\qquad - \tilde{\varphi}(\underline{k})\tilde{\varphi}^*(\underline{k})]\, \tilde{k}^2\, d\underline{k}\end{aligned} \tag{2.4}$$

$$H_{pf} = \sum_{s=1}^{n} \varepsilon_s [\varphi(\underline{r}_s, T) - \underline{\alpha}_s \cdot \underline{A}(\underline{r}_s, T)]. \tag{2.5}$$

When the single equation with common time T is replaced by the set of equations with separate times t_s in accordance with DFP, and after several transformations and use of auxiliary conditions, it is shown in GE III that the equations to be solved are

$$(c\,\underline{\alpha}_s \cdot \underline{P}'_s + m_s c^2 \beta_s)\Omega = T'_s \Omega, \tag{2.6}$$

where

$$\underline{P}'_s = \underline{p}_s - \frac{\varepsilon_s}{c}\,\underline{D}(\underline{r}_s,t_s) - \frac{\varepsilon_s}{2c}\,\nabla_s U_s, \tag{2.7}$$

$$\underline{T}'_s = i\hbar\,\frac{\partial}{\partial t_s} - \frac{\varepsilon_s}{2c}\,\frac{\partial U_s}{\partial t_s} - \frac{1}{4\pi}\,\frac{\varepsilon_s{}^2}{2a}. \tag{2.8}$$

The last two equations are GE III (4.2) and (4.3), with obvious minor changes in notation. The definition of U_s is given by GE III (4.1):

$$U_s \equiv \sum_u{}' \frac{\varepsilon_u}{(2\pi)^3}\int\!\!\int\left[\frac{\sin(\varphi_s - \varphi_u)}{k^3} - \frac{\sin(\tilde{\varphi}_s - \tilde{\varphi}_u)}{\tilde{k}^3}\right] d\underline{k},$$

where

$$\varphi_s = ckt_s - \underline{k}\cdot\underline{r}_s, \qquad \tilde{\varphi}_s = c\tilde{k}t_s - \underline{k}\cdot\underline{r}_s.$$

In order to obtain the wave equation for the system, DFP shows that the equations for the individual particles are to be added, and the times are to be set equal to the common time T. Now it is readily shown that

$$\partial/\partial T = \partial/\partial t + \sum_s \partial/\partial t_s;$$

and inasmuch as Ω is independent of t, the effect of adding equations and setting times equal is to replace $\partial/\partial t_s$ in individual equations by $\partial/\partial T$ in combined equation. The resultant equation is further simplified by showing that $\nabla_s U_s = 0$ when times are set equal, and making use of GE III (4.3) and (4.12), namely

$$\frac{1}{c}\frac{\partial U_s}{\partial t_s} = \sum_u{}' \frac{\varepsilon_u}{4\pi|\underline{r}_s - \underline{r}_u|}\left(1 - e^{-|\underline{r}_s - \underline{r}_u|/a}\right).$$

The resulting wave equation for a system of particles in the generalized quantum electrodynamics is thus[14]

$$\sum_s \{ \underline{\alpha}_s \cdot [c\underline{p}_s - \varepsilon_s \underline{D}(\underline{r}_s, t)] + m_s c^2 \beta_s \} \Omega =$$

$$i\hbar \frac{\partial}{\partial t} - \frac{1}{2} \sum_{s,u}{}' \frac{\varepsilon_s \varepsilon_u}{4\pi |\underline{r}_s - \underline{r}_u|} (1 - e^{-|\underline{r}_s - \underline{r}_u|/a})$$

$$- \frac{1}{4\pi} \sum_s \frac{\varepsilon_s^2}{2a} \quad . \qquad (2.9)$$

Representation of Wave Equation in Functional Formalism[15]

It will be convenient to transform the field variables to their Fourier amplitudes, by means of equation GE II, (3.3):

$$\begin{aligned} \underline{D}(\underline{r}_s, t) = {} & (1/2\pi)^{3/2} \int \{ \underline{D}(\underline{k}) \exp [i(\underline{k} \cdot \underline{r}_s - kct)] \\ & + \underline{D} (\underline{k}) \exp [-i(\underline{k} \cdot \underline{r}_s - kct)] \} d\underline{k} \\ & + (1/2\pi)^{3/2} \int \{ \tilde{\underline{D}}(\underline{k}) \exp [i(\underline{k} \cdot \underline{r}_s - \tilde{k}ct)] \\ & + \tilde{\underline{D}}^* (\underline{k}) \exp [-i(\underline{k} \cdot \underline{r}_s - \tilde{k}ct)] \} d\underline{k}. \end{aligned}$$

From the commutation rules in GE III (3.7)[16], it follows that $\underline{D}(\underline{k})$ may be represented by

$$\underline{D}(\underline{k}) = \sqrt{c\hbar/2k} \sum_{j=1}^{3} \beta_j (1/k) \underline{k} \times \underline{e}_j b(\underline{k}, j) \qquad (2.10)$$

where $\beta_j^2 = 1$, $\underline{e}_j$ are a set of Cartesian unit base vectors,

---

14 This equation was derived earlier by C. Kikuchi in different manner, but has not been previously published.
15 This is the formalism developed by Fock in reference 12.
16 Note the typographical error in the second equation; the tilde over the $\tilde{k}^2$ in right hand side has been omitted.

and the $b(\underline{k},j)$ are operators satisfying

$$\left[b(\underline{k},j),\ b^*(\underline{k}',j')\right] = \delta_{jj'}\,\delta(\underline{k}-\underline{k}'); \tag{2.11}$$

and $\tilde{\underline{D}}(\underline{k})$ by

$$\tilde{\underline{D}}(\underline{k}) = \sqrt{c\hbar/2k}\,\sum_{j=1}^{3}\tilde{\beta}_j(1/a\tilde{k})(a\underline{k}\times\underline{e}_j + \underline{e}_j)\,\tilde{b}(\underline{k},j) \tag{2.12}$$

where ${\beta_j}^2 = 1$, $\underline{e}_j$ are defined previously, and $\tilde{b}(\underline{k},j)$ are operators satisfying

$$\left[\tilde{b}(\underline{k},j),\ \tilde{b}^*(\underline{k}',j')\right] = -\delta_{jj'}\delta(\underline{k}-\underline{k}'). \tag{2.13}$$

Upon definition of

$$G^*(\underline{k},j) \equiv (1/2\pi)^{3/2}\sqrt{c\hbar/2k}\sum_s \varepsilon_s\beta_j(1/k)\underline{\alpha}_s\cdot\underline{k}\ \underline{e}_j\ e^{i(\underline{k}\cdot\underline{r}_s - kct)}, \tag{2.14}$$

$$\tilde{G}^*(\underline{k},j) \equiv (1/2\pi)^{3/2}\sqrt{c\hbar/2\tilde{k}}\sum_s \varepsilon_s\beta_j(1/a\tilde{k})(\underline{\alpha}_s\cdot a\underline{k}\ \underline{e}_j + \underline{\alpha}_s\cdot\underline{e}_j)\ \cdot$$

$$\cdot\ e^{i(\underline{k}\cdot\underline{r}_s - \tilde{k}ct)}, \tag{2.15}$$

and

$$H \equiv \sum_s\left[\underline{\alpha}_s\cdot c\underline{p}_s + m_s c^2\beta_s\right] + \frac{1}{2}\sum_{s,u}{}' \frac{\varepsilon_s\varepsilon_u}{4\pi|\underline{r}_s - \underline{r}_u|}\left(1 - e^{-|\underline{r}_s - \underline{r}_u|/a}\right)$$

$$+ \frac{1}{4\pi}\sum_s \frac{{\varepsilon_s}^2}{2a}, \tag{2.16}$$

the wave equation (2.9) becomes

$$H\Omega - i\hbar\,\partial\Omega/\partial t =$$

$$\left\{\sum_{j=1}^{3}\int d\underline{k}\left[G^*(\underline{k},j)b(\underline{k},j) + G(\underline{k},j)b^*(\underline{k},j)\right.\right.$$

$$\left.\left. + \tilde{G}^*(\underline{k},j)\tilde{b}(\underline{k},j) + \tilde{G}(\underline{k},j)\tilde{b}^*(\underline{k},j)\right]\right\}\Omega. \tag{2.17}$$

The time factors in the exponentials in G's may be eliminated through transformations of type

$$e^{-iwt}b(\underline{k},j)e^{iwt},$$

where

$$w = c\sum_{j'=1}^{3}\int d\underline{k}'[k'b^*(\underline{k}',j')b(\underline{k}',j') - \tilde{k}'\tilde{b}^*(\underline{k}',j')\tilde{b}(\underline{k}',j')]. \tag{2.18}$$

From the commutation rules for b and b, it follows that

$$e^{-iwt}b(\underline{k},j)e^{iwt} = b(\underline{k},j)e^{+ickt},$$

$$e^{-iwt}b^*(\underline{k},j)e^{iwt} = b^*(\underline{k},j)e^{-ickt},$$

$$e^{-iwt}\tilde{b}(\underline{k},j)e^{iwt} = \tilde{b}(\underline{k},j)e^{+ic\tilde{k}t},$$

$$e^{-iwt}\tilde{b}^*(\underline{k},j)e^{iwt} = \tilde{b}^*(\underline{k},j)e^{-ic\tilde{k}t}.$$

There is also the relationship

$$e^{-iwt}(-i\hbar\ \partial/\partial t)e^{iwt} = -i\hbar\ \partial/\partial t + \hbar w. \tag{2.19}$$

Upon definition of

$$G_0^*(\underline{k},j) = G^*(\underline{k},j)e^{ickt}\text{, and so on,}$$

the transformed wave equation becomes (where the transformed functional is designated by the same symbol as the original functional)

$$(H - i\hbar\ \partial/\partial t) + \hbar c\left\{\sum_{j=1}^{3}\int d\underline{k}\,[kb^*(\underline{k},j)b(\underline{k},j) - \tilde{k}\tilde{b}^*(\underline{k},j)\tilde{b}(\underline{k},j)]\right\}\Omega =$$

$$\left\{\sum_{j=1}^{3} d\underline{k}\,[G_0^*(\underline{k},j)b(\underline{k},j) + G_0(\underline{k},j)b^*(\underline{k},j) + \tilde{G}_0^*(\underline{k},j)\tilde{b}(\underline{k},j) + \tilde{G}_0(\underline{k},j)\tilde{b}^*(\underline{k},j)]\right\}\Omega. \tag{2.20}$$

Explicit Representation for Field Operators and Functional

A sufficiently general form[17] for the functional is

$$\Omega \equiv \sum_{r,s} \Omega_{rs}$$

where

$$\Omega_{rs} \equiv \sum_{j_1,\dots j_r} \sum_{i_1,\dots i_s} \int\dots\int d\underline{k}_1\dots d\underline{k}_r \; d\underline{\ell}_1\dots d\underline{\ell}_s \; \cdot$$

$$\cdot \psi_r(\underline{k}_1,j_1\dots\underline{k}_r,j_r)\; \bar{b}(\underline{k}_1,j_1)\dots\bar{b}(\underline{k}_r,j_r) \;\cdot$$

$$\cdot \tilde{\psi}_s(\underline{\ell}_1,i_1\dots\underline{\ell}_s,i_s)\; \bar{\bar{b}}(\underline{\ell}_1,i_1)\dots\bar{\bar{b}}(\underline{\ell}_s,i_s). \qquad (2.21)$$

Each sum is to be taken from 1 to 3 over the values of j and i, and each integral over the entire momentum space of $\underline{k}$ and $\underline{\ell}$. The symbol $\underline{\ell}$, of the same nature as $\underline{k}$, has been introduced temporarily for clarity.

The functional derivative is defined by

$$\frac{\delta\Omega[\bar{b}(\underline{k},j)]}{\delta\bar{b}(\underline{k}',j')} \equiv \lim_{\eta\to 0}\frac{1}{\eta}\Big\{\Omega[\bar{b}(\underline{k},j) + \eta\delta_{jj'}\delta(\underline{k}' - \underline{k})]$$

$$-\Omega[\bar{b}(\underline{k},j)]\Big\}, \qquad (2.22)$$

where $\underline{k},j$ are the variables of integration and summation in the functional.

Definitions (2.21) and (2.22) permit the association

$$b(\underline{k},j) \sim \delta/\delta\bar{b}(\underline{k},j); \quad b^{*}(\underline{k},j) \sim \bar{b}(\underline{k},j);$$

$$b(\underline{\ell},i) \sim \delta/\delta b(\underline{\ell},i); \quad \tilde{b}^{*}(\underline{\ell},i) \sim -\bar{\bar{b}}\,(\underline{\ell},i) \qquad (2.23)$$

---

17 B. Podolsky in a private communication has suggested a notation which is more useful for problems of a general nature; however, for the specific case in this paper the present notation is adequate.

for it may readily be shown that

$$(\delta/\delta\bar{b}(\underline{k},j))\,\bar{b}(\underline{k}',j')\Omega - \bar{b}(\underline{k}',j')(\delta/\delta\bar{b}(\underline{k},j))\Omega = \delta_{jj'}\delta(\underline{k}-\underline{k}')\Omega$$

and (2.24)

$$(-\delta/\delta\bar{\bar{b}}(\underline{\ell},i))\,\bar{\bar{b}}(\underline{\ell}',i')\Omega - \bar{\bar{b}}(\underline{\ell}',i')(-\delta/\delta\bar{\bar{b}}(\underline{\ell},i))\Omega = -\delta_{ii'}\delta(\underline{\ell}-\underline{\ell}')\Omega .$$

These two equations are to be compared with the commutation rules (2.11) and (2.13).

Application to Wave Equation

The explicit representation developed in the preceding section may be applied to the wave equation (2.20) in order to obtain an ordinary wave equation in $\underline{k}$-space. For the immediate purpose of this paper, we are interested in the case where the series of functionals is to be broken off after only the first three terms. Then

$$\Omega = \Omega_{00} + \Omega_{10} + \Omega_{01},$$

where

$$\Omega_{00} = \psi_0\tilde{\psi}_0$$

$$\Omega_{10} = \tilde{\psi}_0\sum_{j_1}\int d\underline{k}_1\psi_1(\underline{k}_1,j_1)\,\bar{b}(\underline{k}_1,j_1) = \tilde{\psi}_0\sum_j\int d\underline{k}\psi_1(\underline{k},j)\,\bar{b}(\underline{k},j)$$

$$\Omega_{01} = \psi_0\sum_{i_1}\int d\underline{\ell}_1\tilde{\psi}_1(\_{}_1,i_1)\,\bar{\bar{b}}(\underline{\ell}_1,i_1) = \psi_0\sum_j\int d\underline{k}\tilde{\psi}_1(\underline{k},j)\,\bar{\bar{b}}(\underline{k},j).$$

The substitution of this expression in the wave equation after considerable computation results in the following three equations:

$$(H - i\hbar\ \partial/\partial t)\psi_0\tilde{\psi}_0 =$$

$$\sum_j \int d\underline{k}\ G_0^* (\underline{k},j)\tilde{\psi}_0\psi_1(\underline{k},j) + \sum_j \int d\underline{k}\ \tilde{G}_0^* (\underline{k},j)\psi_0\tilde{\psi}_1(\underline{k},j) \quad (2.25)$$

$$(H + \hbar ck - i\hbar\ \partial/\partial t)\tilde{\psi}_0\psi_1(\underline{k},j) = G_0(\underline{k},j)\ \tilde{\psi}_0\psi_0 \quad (2.26)$$

$$(H + \hbar c\tilde{k} - i\hbar\ \partial/\partial t)\psi_0\tilde{\psi}_1(\underline{k},j) = -\tilde{G}_0(\underline{k},j)\ \tilde{\psi}_0\psi_0. \quad (2.27)$$

RELATIVISTIC INTERACTION OF TWO ELECTRONS

Suppose the system consists only of two electrons and their field. To obtain a first-order approximation (i.e., matrix elements proportional to the square of the charge on the electron) it is possible to treat the last two terms in the definition of H (2.16) and the entire right hand side of (2.25) as perturbations of an unperturbed Hamiltonian consisting of the first two terms of H. In order to eliminate ψ_1 and $\tilde{\psi}_1$ on the right hand side of (2.25), the two succeeding equations are solved for these two functions, with $\psi_0\tilde{\psi}_0$ approximated by the wave function for the unperturbed system. The substitution of these results into (2.25) provides an equation amenable to methods of standard perturbation theory.

Wave Function for the Unperturbed System

The representative for two free electrons with momenta $\underline{p}_1{}^{o}$ and $\underline{p}_2{}^{o}$, and signs of energy and direction of spin designated by $s_1{}^{o}$ and $s_2{}^{o}$, is given in $\underline{r}_1, \underline{r}_2, \zeta_1, \zeta_2$ space by

$$(\underline{r}_1, \zeta_1;\ \underline{r}_2, \zeta_2 | \underline{p}_1{}^{o}, s_1{}^{o};\ \underline{p}_2{}^{o}, s_2{}^{o}) = e^{-iW^{o}t/\hbar}\,\varphi_o^{o} \quad (3.1)$$

where

$$\varphi_o^o = (1/2\pi\hbar)^3 e^{-i(\underline{p}_1{}^o\cdot\underline{r}_1 + \underline{p}_2{}^o\cdot\underline{r}_2)/\hbar} u_{\zeta_1\zeta_2}(\underline{p}_1{}^o s_1{}^o; \underline{p}_2{}^o s_2{}^o) \qquad (3.2)$$

Here u is antisymmetric, and has sixteen components corresponding to variables ζ_1, ζ_2, each of which has four variables. The representative (3.1) is a solution of the wave equation

$$(F_1 + F_2)\psi = W^o\psi,$$

where

$$F_s \equiv \underline{\alpha}_s\cdot c\underline{p}_s + m_s c^2\beta_s \qquad (3.3)$$

and

$$W^o = W_1^o + W_2^o , \qquad (3.4)$$

with

$$F_1\varphi_o^o = W_1^o\varphi_o^o , \quad F_2\varphi_o^o = W_2^o\varphi_o^o . \qquad (3.5)$$

<u>Elimination of ψ_1 and $\tilde{\psi}_1$</u>

Let us define

$$\tilde{\psi}_o\psi_1 \equiv f \equiv f_o e^{-iW^o t/\hbar} , \quad \psi_o\tilde{\psi}_1 \equiv g \equiv g_o e^{-iW^o t/\hbar} \qquad (3.6)$$

It is clear from (2.26) and (2.27) that f and g have the same time dependence as (3.1), when $\tilde{\psi}_o\psi_o$ has been approximated by this wave function. Hence f_o and g_o are independent of time, and (2.26) and (2.27) lead to

$$(F_1 + F_2 + \hbar ck - W^o)f^o = G_o(\underline{k}, j)\varphi_o^o \qquad (3.7)$$

$$(F_1 + F_2 + \hbar c\tilde{k} - W^o)g^o = -\tilde{G}_o(\underline{k}, j)\varphi_o^o \qquad (3.8)$$

where the static interaction and the self-energy have been neglected, as they are proportional to the square of the charge, and would give terms proportional to powers higher than second in the final interaction matrix element.

The solutions for f^{o} and g^{o} are

$$f^{o} = \theta \varphi_{o}^{o}, \quad g^{o} = \tilde{\theta} \varphi_{o}^{o} \quad , \tag{3.9}$$

where

$$\theta \equiv (1/2\pi)^{3/2} \sqrt{\hbar c/2k} \quad \cdot$$

$$\cdot \left\{ (F_1 + \hbar ck - W_o^1)^{-1} \varepsilon_1 \beta_j (1/k)(\underline{\alpha}_1 \cdot \underline{k} \times \underline{e}_j) \cdot e^{-i\underline{k}\cdot\underline{r}_1} \right.$$

$$\left. + (F_2 + \hbar ck - W_o^2)^{-1} \varepsilon_2 \beta_j (1/k)(\underline{\alpha}_2 \cdot \underline{k} \times \underline{e}_j) e^{-i\underline{k}\cdot\underline{r}_2} \right\} \tag{3.10}$$

$$\tilde{\theta} \equiv (1/2\pi)^{3/2} \sqrt{\hbar c/2\tilde{k}} \quad \cdot$$

$$\cdot \left\{ (F_1 + \hbar ck - W_o^1)^{-1} \varepsilon_1 \tilde{\beta}_j (1/a\tilde{k})(\underline{\alpha}_1 \cdot a\underline{k} \times \underline{e}_j + \underline{\alpha}_1 \cdot \underline{e}_j) e^{-i\underline{k}\cdot\underline{r}_1} \right.$$

$$\left. + (F_2 + \hbar ck - W_o^2)^{-1} \varepsilon_2 \tilde{\beta}_j (1/a\tilde{k})(\underline{\alpha}_2 \cdot a\underline{k} \times \underline{e}_j + \underline{\alpha}_2 \cdot \underline{e}_j) e^{-i\underline{k}\cdot\underline{r}_2} \right\} \tag{3.11}$$

Here we have made use of the fact that $\underline{\alpha}_1$ and $\underline{\alpha}_2$ commute, and that φ_o^o satisfies (3.5).

The wave equation (2.25) becomes

$$(H - i\hbar\, \partial/\partial t)\psi_o \tilde{\psi}_o = \sum_j \int d\underline{k} \left[G_o^* \, \theta - \tilde{G}_o^* \, \tilde{\theta}\right] \psi \tag{3.12}$$

Now ψ differs from $\psi_o \tilde{\psi}_o$ only by a perturbation contribution; and the operator preceding Ψ is itself a perturbation operator. Hence we may replace Ψ by $\Psi_o\tilde{\Psi}_o$, and the equation is in standard form for application of perturbation theory.

<u>Calculation of Interaction Matrix Element</u>

The perturbing energies are, for two particles of charge ε_1 and ε_2,

$$\varepsilon_1\varepsilon_2(1 - e^{-|\underline{r}_1 - \underline{r}_2|/a})/4\pi|\underline{r}_1 - \underline{r}_2| \equiv U_1 \qquad (3.13)$$

$$(\varepsilon_1{}^2 + \varepsilon_2{}^2)/8\pi a \equiv V_1 \qquad (3.14)$$

$$-\int d\underline{k} \sum_j (G_o^* \, \theta - \tilde{G}_o^* \, \tilde{\theta}) \equiv U_2 + V_2, \qquad (3.15)$$

where U_2 represents the part of the left-hand side of (3.15) containing terms in $\varepsilon_1\varepsilon_2$, and V_2 represents the part containing terms in $\varepsilon_1{}^2$ and $\varepsilon_2{}^2$. Since we are interested in the interaction only, we shall calculate only the matrix elements for U_1 and U_2. Further we assume conservation of energy: $W_1 + W_2 = W_1^o + W_2^o$. It is well known that the matrix element

$$(\underline{p}_1, s_1; \underline{p}_2, s_2|\varepsilon_1\varepsilon_2/4\pi|\underline{r}_1 - \underline{r}_2||\underline{p}_1^o, s_1^o; \underline{p}_2^o, s_2^o) =$$

$$(1/2\pi)^3\varepsilon_1\varepsilon_2 \; (\underline{P}_1 + \underline{P}_2)(1/\hbar P_1{}^2)(u^*, u^o), \qquad (3.16)$$

where $\underline{P}_s \equiv \underline{p}_s - \underline{p}_s^o$, and

$$(u^*, u^o) \equiv \sum_{\rho_1\rho_2} u_{\rho_1\rho_2} \; (\underline{p}_1, s_1; \underline{p}_2, s_2) u_{\rho_1\rho_2} \; (\underline{p}_1^o, s_1^o; \underline{p}_2^o, s_2^o).$$

From the $-\epsilon_1\epsilon_2 e^{-|\underline{r}_1 - \underline{r}_2|/a}/4\pi|\underline{r}_1 - \underline{r}_2|$ term, the contribution to the interaction matrix element is calculated as the negative of (3.16), with $P_1{}^2$ replaced by $P_1{}^2 + \hbar^2/a^2$.

For the part of $-\int d\underline{k} \sum_j G_o^* \theta$ which contains $\epsilon_1\epsilon_2$, the contribution is

$$(1/2\pi)^3\epsilon_1\epsilon_2\delta(\underline{P}_1 + \underline{P}_2)\{\hbar[P_1{}^2 - (W_1 - W_1^o)^2/c^2]\}^{-1} \cdot$$
$$(u^*,(\underline{\alpha}_1\cdot\underline{\alpha}_2 - \underline{\alpha}_1\cdot\underline{P}_1\underline{\alpha}_2\cdot\underline{P}_1/P_1{}^2)u^o) \; ; \tag{3.17}$$

for the part of $+\int d\underline{k} \sum_j \tilde{G}_o^* \tilde{\theta}$ which contains $\epsilon_1\epsilon_2$, the contribution is given by the negative of (3.17) with $P_1{}^2$ replaced by $P_1{}^2 + h^2/a^2$.

After some computation, we get for the interaction element the following:

$$(1/2\pi)^3(1/\hbar)\delta(\underline{p}_1 + \underline{p}_2 - \underline{p}_1^o - \underline{p}_2^o) \cdot$$
$$\cdot\{|\underline{p}_1 - \underline{p}_1^o|^2 - (W_1 - W_1^o)/c^2\}^{-1}\{1 + (a^2/\hbar^2)[|p_1 - p_1^o|^2$$
$$- (W_1 - W_1^o)/c^2]\}^{-1} (u^*,(1 - \underline{\alpha}_1\cdot\underline{\alpha}_2)u^o). \tag{3.18}$$

This is a generalization of Møller's formula.[18] It will be noticed that it is relativistically invariant, and reduces to Møller's expression as $a \to 0$.

---

18 C. Møller, Zeits. f. Physik, 70, 786 (1931).

APPENDIX

Proof that Relations (2.10) and (2.12) are consistent with Commutation Relations GE III (3.7)

According to (2.10) and (2.11),

$$[D_\ell(\underline{k}),D_m^*(\underline{k}')] =$$

$$\sqrt{c\hbar/2k}\,\sqrt{c\hbar/2k'}\sum_{j,j'}\beta_j\beta_{j'}(1/k)(1/k')(\underline{k}\times\underline{e}_j)\cdot\underline{e}_\ell\;(\underline{k}\times\underline{e}_j')\cdot\underline{e}_m$$

$$[b(\underline{k},j),b^*(\underline{k},j)]$$

$$= (c\hbar/2k^3)\sum_{j,j'}\beta_j\beta_{j'}(\underline{k}\times\underline{e}_\ell\cdot\underline{e}_j)(\underline{k}\times\underline{e}_m\cdot\underline{e}_j)\;\delta_{jj'}\delta(\underline{k}-\underline{k}')$$

$$= (c\hbar/2k^3)(\underline{k}\times\underline{e}_\ell)\cdot(\underline{k}\times\underline{e}_m)\;\delta(\underline{k}-\underline{k}')$$

$$= (c\hbar/2k^3)(\underline{k}\cdot\underline{k}\;\underline{e}_\ell\cdot\underline{e}_m - \underline{k}\cdot\underline{e}_\ell\;\underline{k}\cdot\underline{e}_m)\,\delta(\underline{k}-\underline{k}')$$

$$= (c\hbar/2k)(\delta_{\ell m} - k_\ell k_m/k^2)\;\delta(\underline{k}-\underline{k}'),$$

which is the same as GE III (3.7).

According to (2.12) and (2.13),

$$[\tilde{D}_\ell(\underline{k}),\tilde{D}_m^*(\underline{k}')] =$$

$$\sqrt{c\hbar/2\tilde{k}}\,\sqrt{c\hbar/2\tilde{k}'}\sum_{j,j'}\tilde{\beta}_j\tilde{\beta}_{j'}(1/a\tilde{k})(1/a\tilde{k}')$$

$$(a\underline{k}\times\underline{e}_j + \underline{e}_j)\cdot\underline{e}_\ell\;(a\underline{k}\times\underline{e}_j)\cdot\underline{e}_m\;[b(\underline{k},j),b^*(\underline{k},j)]$$

$$= -(c\hbar/2a^2\tilde{k}^3)\sum_{j,j'}\tilde{\beta}_j\tilde{\beta}_{j'}(a\underline{k}\times\underline{e}_\ell - \underline{e}_\ell)\cdot\underline{e}_j\;(a\underline{k}\times\underline{e}_m - \underline{e}_m)\cdot\underline{e}_j$$

$$\delta_{jj'}\delta(\underline{k}-\underline{k}')$$

$$= -(c\hbar/2a^2\tilde{k}^3)(a\underline{k}\times\underline{e}_\ell - \underline{e}_\ell)\cdot(a\underline{k}\times\underline{e}_m - \underline{e}_m)\,\delta(\underline{k} - \underline{k}').$$

Now the dot product becomes

$$(a\underline{k}\times\underline{e}_\ell)\cdot(a\underline{k}\times\underline{e}_m) + \underline{e}_\ell\cdot\underline{e}_m - a\underline{k}\times\underline{e}_\ell\cdot\underline{e}_m - a\underline{k}\ \underline{e}_m\cdot\underline{e}_\ell$$

$$= a^2\underline{k}\cdot\underline{k}\ \underline{e}_\ell\cdot\underline{e}_m - a^2\underline{k}\cdot\underline{e}_\ell\ \underline{k}\cdot\underline{e}_m + \underline{e}_\ell\cdot\underline{e}_m$$

$$= a^2\tilde{k}^2\delta_{\ell m} - a^2k_\ell k_m.$$

Then

$$[\tilde{D}_\ell(\underline{k}), \tilde{D}_m^*(\underline{k}')] = -(c\hbar/2\tilde{k})(\delta_{\ell m} - k_\ell k_m/\tilde{k}^2)\,\delta(\underline{k} - \underline{k}')$$

which is the same as GE III (3.7).

Computation of Matrix Element for U_1 (3.13)

$$(\underline{p}_1,s_1;\underline{p}_2,s_2|-\varepsilon_1\varepsilon_2 e^{-|\underline{r}_1-\underline{r}_2|/a}/4\pi|\underline{r}_1-\underline{r}_2|\,|\underline{p}_1^o,s_1^o;\underline{p}_2^o,s_2^o) =$$

$$-\sum_{\zeta_1\zeta_2}\int d\underline{r}_1\int d\underline{r}_2\varphi_o^*\{\varepsilon_1\varepsilon_2 e^{-|\underline{r}_1-\underline{r}_2|/a}/4\pi|\underline{r}_1-\underline{r}_2|\}\varphi_o^o =$$

$$-\varepsilon_1\varepsilon_2(u^*,u^o)(1/2\pi\hbar)^6\iint d\underline{r}_1 d\underline{r}_2 e^{-|\underline{r}_1-\underline{r}_2|/a}\cdot$$

$$\cdot\, e^{-i[(\underline{p}_1-\underline{p}_1^o)\cdot(\underline{r}_1-\underline{r}_2+\underline{r}_2)+(\underline{p}_2-\underline{p}_2^o)\cdot\underline{r}_2]/\hbar}/|\underline{r}_1-\underline{r}_2|$$

Define $\underline{P}_1\equiv\underline{p}_1-\underline{p}_1^o$, $\underline{P}_2\equiv\underline{p}_2-\underline{p}_2^o$, $\underline{R}\equiv\underline{r}_1-\underline{r}_2$; the integral immediately above becomes

$$\iint d\underline{r}_2 d\underline{R}\, e^{-i[\underline{P}_1\cdot\underline{R}+(\underline{P}_1+\underline{P}_2)\cdot\underline{r}_2]/\hbar}(e^{-R/a}/R) =$$

$$\int d\underline{r}_2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}e^{-(i/\hbar)P_1R\cos\theta}(e^{-R/a}/R)R^2\sin\theta\, dR\,d\theta\,d\phi =$$

$$(2\pi\hbar)^3\delta(\underline{P}_1+\underline{P}_2)(4\pi\hbar/P_1)\cdot\int_0^{\infty}\sin(P_1R/\hbar)\cdot e^{-R/a}dR =$$

$$(2\pi\hbar)^3\delta(\underline{P}_1+\underline{P}_2)(4\pi\hbar/P_1)\cdot\frac{P_1/\hbar}{1/a^2+P_1^2/\hbar^2}$$

Then the matrix element is finally

$$-(1/2\pi)^3\varepsilon_1\varepsilon_2\delta(\underline{P}_1+\underline{P}_2)(1/\hbar)[P_1^2+\hbar^2/a^2]^{-1}\cdot(u^*,u^o).$$

<u>Computation of U_2 (3.15)</u>

We have to evaluate $-\int d\underline{k} \sum_j (G_o^* \theta - \tilde{G}_o^* \tilde{\theta})$.

From the appropriate definitions (2.14, following 2.19, and 3.10), we have

$$\sum_j G_o^* \theta = \sum_j (1/2\pi)^3 (\hbar c/2k^3)$$

$$\{\epsilon_1{}^2 \underline{\alpha}_1 \cdot \underline{k}\times\underline{e}_j e^{-i\underline{k}\cdot\underline{r}_1}(F_1 + \hbar ck - W_1^o)^{-1} \underline{\alpha}_1 \cdot \underline{k}\times\underline{e}_j e^{-i\underline{k}\cdot\underline{r}_1}$$

$$+ \epsilon_2{}^2 \underline{\alpha}_2 \cdot \underline{k}\times\underline{e}_j e^{-i\underline{k}\cdot\underline{r}_1}(F_2 + \hbar ck - W_2^o)^{-1} \underline{\alpha}_2 \cdot \underline{k}\times\underline{e}_j e^{-i\underline{k}\cdot\underline{r}_2}$$

$$+ \epsilon_1\epsilon_2 \underline{\alpha}_2 \cdot \underline{k}\times\underline{e}_j (F_1 + \hbar ck - W_1^o)^{-1} \underline{\alpha}_1 \cdot \underline{k}\times\underline{e}_j e^{i\underline{k}\cdot(\underline{r}_2-\underline{r}_1)}$$

$$+ \epsilon_1\epsilon_2 \underline{\alpha}_1 \cdot \underline{k}\times\underline{e}_j (F_2 + \hbar ck - W_2^o)^{-1} \underline{\alpha}_2 \cdot \underline{k}\ \underline{e}_j e^{i\underline{k}\cdot(\underline{r}_2-\underline{r}_1)}\}.$$

We are interested only in the terms in $\epsilon_1\epsilon_2$; the matrix element for the first of these is, since F_1 is Hermitian,

$$\sum_j \sum \int \epsilon_1\epsilon_2 \overline{(F_1 + \hbar ck - W_1^o)^{-1} \varphi_o^*} \underline{\alpha}_2 \cdot \underline{k}\times\underline{e}_j\ \underline{\alpha}_1 \cdot \underline{k}\times\underline{e}_j\ e^{-i\underline{k}\cdot(\underline{r}_2-\underline{r}_1)} \varphi_o^o$$

where the second sum is to be taken over the spin variables and the integrals over the $\underline{r}_1, \underline{r}_2$ variables. With the help of (3.5), the expression becomes

$$\epsilon_1\epsilon_2 \sum \int (W_1 + \hbar ck - W_1^o)^{-1} \varphi_o^* (\underline{\alpha}_1 \times \underline{k})\cdot(\underline{\alpha}_2 \times \underline{k}) e^{i\underline{k}\cdot(\underline{r}_2-\underline{r}_1)} \varphi_o^o =$$

$$\epsilon_1\epsilon_2 \int (W_1 + \hbar ck - W_1^o)^{-1} e^{-i(\underline{p}_1\cdot\underline{r}_1 + \underline{p}_2\cdot\underline{r}_2)/\hbar}$$

$$(u^*, (\underline{\alpha}_1 \cdot \underline{\alpha}_2 k^2 - \underline{\alpha}_1 \cdot \underline{k}\ \underline{\alpha}_2 \cdot \underline{k}) e^{i\underline{k}\cdot(\underline{r}_2-\underline{r}_1)} e^{i(\underline{p}_1^o\cdot\underline{r}_1 + \underline{p}_2^o\cdot\underline{r}_2)/\hbar}.$$

The complete expression for the term without tildes is

$$-(1/2\pi)^3(\hbar c/2k)\,\varepsilon_1\varepsilon_2\int d\underline{k}\int d\underline{r}_1 d\underline{r}_2 (W_1 + \hbar ck - W_1^o)^{-1}$$

$$e^{-i(\underline{p}_1-\hbar\underline{k})\cdot\underline{r}_1/\hbar}e^{-i(\underline{p}_2+\hbar\underline{k})\cdot\underline{r}_2/\hbar}(u^*,(\underline{\alpha}_1\cdot\underline{\alpha}_2 - \underline{\alpha}_1\cdot\underline{k}\;\underline{\alpha}_2\cdot\underline{k}/k^2)).$$

Integration with respect to $\underline{r}_1$ gives a $\delta(\underline{p}_1 - \hbar\underline{k})$ factor, and integration with respect to $\hbar\underline{k}$ replaces the $\hbar\underline{k}$ by $\underline{p}_1$. The final integration with respect to $\underline{r}_2$ gives a $\delta(\underline{p}_2 + \underline{p}_1)$ factor. The result is

$$(1/2\pi)^3\varepsilon_1\varepsilon_2(c/\hbar)\delta(\underline{p}_1 + \underline{p}_2)(W_1 - W_1^o + c|\underline{p}_1 - \underline{p}_1^o|)^{-1}$$

$$(2|\underline{p}_1 - \underline{p}_1^o|^3)^{-1}\;(u^*,(\underline{\alpha}_1\cdot\underline{\alpha}_2 - \underline{\alpha}_1\cdot\underline{p}_1\;\underline{\alpha}_2\cdot\underline{p}_1)u^o)$$

For the second term containing $\varepsilon_1\varepsilon_2$, the subscripts 1 and 2 and interchanged. Upon use of the conservation of energy, we find the sum of the two is

$$-(1/2\pi)^3\varepsilon_1\varepsilon_2\delta(\underline{p}_1 + \underline{p}_2)\left\{\hbar\left[p_1{}^2 - (W_1 - W_1^o)^2/c^2\right]\right\}^{-1}$$

$$(u^*,(\underline{\alpha}_1\cdot\underline{\alpha}_2 - \underline{\alpha}_1\cdot\underline{p}_1\;\underline{\alpha}_2\cdot\underline{p}_1/p_1{}^2)).$$

The calculations for $+\int d\underline{k}\sum_j \tilde{G}_o^*\tilde{\Theta}$ are of the same type.

The part containing the Dirac matrices can be simplified further: for by (3.2) and (3.3)

$$F_1\varphi_o^o = (1/2\pi\hbar)^3 e^{i\underline{p}_2^o\cdot\underline{r}_2/\hbar}.(\underline{\alpha}_1\cdot c\underline{p}_1 e^{i\underline{p}_1^o\cdot\underline{r}_1/\hbar}u^o$$

$$+ m_1c^2\beta_1 e^{i\underline{p}_1^o\cdot\underline{r}_1/\hbar}u^o)$$

$$= (1/2\pi\hbar)^3 e^{i\underline{p}_2^o\cdot\underline{r}_2/\hbar}.(e^{i\underline{p}_1^o\cdot\underline{r}_1/\hbar}\underline{\alpha}_1\cdot c\underline{p}_1 u^o$$

$$+ \underline{\alpha}_1\cdot c\underline{p}_1^o e^{i\underline{p}_1^o\cdot\underline{r}_1/\hbar}u^o + m_1c^2\beta_1 e^{i\,\underline{p}_1^o\cdot\underline{r}_1/\hbar}u^o)$$

$$= (1/2\pi\hbar)^3 e^{i(\underline{p}_1^o\cdot\underline{r}_1+\underline{p}_2^o\cdot\underline{r}_2)/\hbar}(F_1 + \underline{\alpha}_1\cdot c\underline{p}_1^o)u^o$$

$$= W_1^o\varphi_o^o$$

whence

$$(F_1 + \underline{\alpha}_1\cdot c\underline{p}_1^o)u^o = W_1^o u^o;$$

and since, in the expression (3.3) for F_1,

$$\underline{\alpha}_1\cdot c\underline{p}_1 u^o \equiv \underline{\alpha}_1\cdot(\hbar/i)\partial u^o/\partial\underline{r}_1 \equiv 0,$$

it follows that $\underline{\alpha}_1\cdot\underline{p}_1^o u^o = (1/c)(W_1^o - m_1c^2\beta_1)u^o$.

Analogous relations hold for $\underline{\alpha}_1\cdot\underline{p}_1$, $\underline{\alpha}_2\cdot\underline{p}_2^o$, $\underline{\alpha}_2\cdot\underline{p}_2$.

Now $(u^*, -\underline{\alpha}_1\cdot\underline{P}_1\underline{\alpha}_2\cdot\underline{P}_1 u^o) = (u^*, [\underline{\alpha}_1\cdot(\underline{p}_1 - \underline{p}_1^o)\ \underline{\alpha}_2\cdot(\underline{p}_2 - \underline{p}_2^o)]u^o)$, by definition of $\underline{P}_1$ and conservation of momentum, according to $\delta(\underline{P}_1 + \underline{P}_2)$ factor in the matrix element. The expansion of the quantity in brackets gives

$$\underline{\alpha}_1\cdot\underline{p}_1\ \underline{\alpha}_2\cdot\underline{p}_2 - \underline{\alpha}_1\cdot\underline{p}_1^o\ \underline{\alpha}_2\cdot\underline{p}_2 - \underline{\alpha}_1\cdot\underline{p}_1\ \underline{\alpha}_2\cdot\underline{p}_2^o + \underline{\alpha}_1\cdot\underline{p}_1^o\ \underline{\alpha}_2\cdot\underline{p}_2^o.$$

Then $c^2(u^*, \underline{\alpha}_1 \cdot \underline{p}_1\ \underline{\alpha}_2 \cdot \underline{p}_2\ u^o) = ([\underline{\alpha}_2 \cdot \underline{p}_2\ \underline{\alpha}_1 \cdot \underline{p}_1\ u]^*, u)c^2$

$$= ([\underline{\alpha}_2 \cdot \underline{p}_2(W_1 - m_1c^2\beta_1)u]^*, u^o)\ c$$

$$= ([(W_2 - m_2c^2\beta_2)(W_1 - m_1c^2\beta_1)u]^*, u^o)$$

$$= (u^*, (W_2 - m_2c^2\beta_2)(W_1 - m_1c^2\beta_1)u)$$

and

$$-c^2(u^*, \underline{\alpha}_1 \cdot \underline{p}_1^o\ \underline{\alpha}_2 \cdot \underline{p}_2 u^o) = -([\underline{\alpha}_2 \cdot \underline{p}_2 u]^*, \underline{\alpha}_1 \cdot \underline{p}_1^o u^o)\ c^2$$

$$= -([(W_2 - m_2c^2\beta_2)u]^*, (W_1^o - m_1c^2\beta_1)u^o)$$

$$= -(u^*, (W_2 - m_2c^2\beta_2)(W_1^o - m_1c^2\beta_1)u^o).$$

Analogous expressions are computed in similar fashion for the other two terms in the expansion. Upon combining the four terms, the β's disappear, and the result is

$$(u^*, (W_1 - W_1^o)(W_2 - W_2^o)u).$$

By conservation of energy, $W_1 - W_1^o = -(W_2 - W_2^o)$, whence

$$(u^*, -\underline{\alpha}_1 \cdot \underline{P}_1\ \underline{\alpha}_2 \cdot \underline{P}_1 u^o) = -(1/c^2)(u^*, (W_1 - W_1^o)^2 u^o).$$

This is useful in the final computation for (3.18).

Proof of Transformation following Equation (2.18)

We shall prove the first relation

$$e^{-iwt}b(\underline{k},j)e^{iwt} = b(\underline{k},j)e^{+ickt};$$

the others follow in the same manner.

It is clear that w, as defined in (2.18), commutes with $b(\underline{k},j)$ except in the neighborhood of $\underline{k}' = \underline{k}$, for the values of $j' \neq j$, and for all the b's; hence

$$e^{-iwt}b(\underline{k},j)e^{iwt} = \exp\left[-ict\int_{\Delta k} d\underline{k}'k'b^*(\underline{k}',j)b(\underline{k}',j)\right]\cdot$$

$$\cdot\, b(\underline{k},j)\exp\left[+ict\int_{\Delta k} d\underline{k}'k'b^*(\underline{k}',j)b(\underline{k}',j)\right]$$

$$= \exp\left[-ictk\,\Delta k\, b^*(\underline{k},j)b(\underline{k},j)\right].b(\underline{k},j)\,\cdot$$

$$\cdot\, \exp\left[+ictk\,\Delta k\, b^*(\underline{k},j)b(\underline{k},j)\right]$$

$$= \exp\left[-ickt\,\Delta k\, b^*b\right].b.$$

$$\left[1 + ickt\,\Delta k\, b^*b + (ickt\,\Delta k)^2 b^*b\, b^*b/2! + \ldots\right]$$

$$= \exp\left[-ickt\,\Delta k\, b^*b\right].$$

$$\left[b + ickt\,\Delta k\, bb^*\cdot b + (ickt\,\Delta k)^2 bb^*b\, b^*\cdot b/2! + \ldots\right]$$

$$= \exp\left[-ickt\,\Delta k\, b^*b\right]\exp\left[ickt\,\Delta k\, bb^*\right]\cdot b$$

$$= \exp\left[ickt\,\Delta k\,(bb^* - b^*b)\right]\cdot b$$

But $\int [b(\underline{k},j),b^*(\underline{k}',j)]d\underline{k} = \int_{\Delta k}[b(\underline{k},j),b^*(\underline{k}',j)]d\underline{k}$

$$= \Delta k\left\{b(\underline{k},j)b^*(\underline{k},j) - b^*(\underline{k},j)b(\underline{k},j)\right\} = \int \delta(\underline{k} - \underline{k}')d\underline{k} = 1$$

Hence

$$e^{-iwt}b(\underline{k},j)e^{iwt} = b(\underline{k},j)e^{+ickt}.$$